P 11 Energy corrected finite elements for optimal control problems on domains with re-entrant corners (O. Steinbach, B. Wohlmuth) $\rightarrow$ NS, IS

State of the art. It is well-known that standard finite element discretizations of elliptic model problems on two-dimensional domains with re-entrant corners on a sequence of uniformly refined meshes do not guarantee optimal order convergence. In the presence of re-entrant corners with interior angle $\pi < \omega < 2\pi$ the solution will, in general, have singular components of type $r^{\pi/\omega}$ even when the data are smooth; here, r denotes the distance to the singular corners. As a consequence, only reduced convergence rates are obtained by standard finite element methods on quasi-uniform meshes, see, e.g., [6]. Various approaches to recover the full convergence order for finite element methods have been considered in the literature. Quite often methods based on local refinement, see, e.g., [3, 4, 5, 14, 15], are used. An alternative approach is based on the enrichment of the finite element space by suitable singular functions [11, 20]. However both approaches result in global modifications. In order to restore the full convergence order in weighted norms, we consider in this project energy corrected finite elements which are based on *local modifications* of the discrete problems using the techniques introduced in [21]; see also [17, 19] and analyzed in [10, 18]. The required modifications are *local*, i.e., they only affect degrees of freedom associated to an $\mathcal{O}(h)$ neighborhood of the singular point. As a consequence, the structure and all but a finite number of entries in the stiffness matrix remain unchanged which makes it extremely attractive in case of many re-entrant corners.

Thesis project to be supervised by B. Wohlmuth. In this project thesis, we now generalize energy corrected finite elements to optimal control problems with main focus on boundary control and to the Stokes system. However, in a first step we will also investigate the standard elliptic Laplace operator based PDE setting. For previous work on numerical analysis of Dirichlet boundary control problems, we refer to, e.g., [1, 2, 7, 8, 9, 13, 16]. While convex domains are well studied, the theory for the non-convex case is less understood and many open questions exist. We will provide an a priori error analysis restoring optimal convergence order and removing the pollution effect resulting from the the re-entrant corners. To do so, we have to consider the singular components, e.g., for the Stokes system and introduce locally defined corrections depending on the interior angle and the used finite element type, e.g. Taylor-Hood or MINI. For the formulation of the optimal control problem, we assume that the desired state $\bar{u} \in L^2(\Omega_{\text{obs}})^2$ for the observation domain $\Omega_{\text{obs}} \subset \Omega$ and the right-hand side $f \in L^2(\Omega)^2$, which prescribes a given force acting on the fluid. As a model problem, we consider an optimal Dirichlet boundary control problem for the Stokes equations, see, e.g., [12], which is given as follows: Minimize the cost functional

$$\mathcal{J}(u, z) := \frac{1}{2}\|u - \bar{u}\|^2_{L^2(\Omega_{\text{obs}})} + \frac{1}{2}\varrho\|z\|^2_{*},$$

where $\|\cdot\|_*$ is a suitable norm restricted to the control boundary Γ_C, subject to the constraint

$$\begin{aligned} -\nu\Delta u + \nabla p &= f \quad \text{in } \Omega, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega, \\ u &= z \quad \text{on } \Gamma_\mathrm{C}, \\ u &= g \quad \text{on } \Gamma_\mathrm{D}, \\ \nu(\nabla u)n - p\,n &= 0 \quad \text{on } \Gamma_\mathrm{N}, \end{aligned}$$

and the additional control constraints $z_a \leq z \leq z_b$ a.e. on Γ_C. As standard $\Gamma_\mathrm{C}, \Gamma_\mathrm{N}$ and Γ_D form a partition of the boundary $\partial\Omega$. Of special interest for us are domains with multiple re-entrant corners, as they appear in several applications for optimal control problems, such as shown in the following figure.

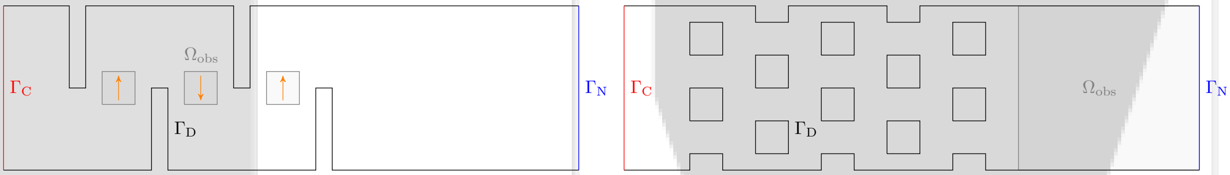

Figure 1: Domains of interest with multiple re-entrant corners

These ideas can be also applied to other control settings, such as Neumann boundary control or distributed control.

A PhD candidate for such a project needs to have a strong background in numerical analysis as well as regularity theory and weighted Sobolev spaces. But also a strong interest in implementational aspects and performance considerations is required.

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